Linear program for partial set cover problem

Consider the following linear program for set cover problem:

$$\begin{aligned} & min \sum_{S} w_s x_s \\ \text{s.t.} \quad & \sum_{S: e \in S} x_s \geq 1 \quad \forall e \in E \\ & x_S \geq 0 \qquad \forall S \in \{S_1, ..., S_m\} \end{aligned}$$

An analog of the above LP for a partial cover problem would be

$$\begin{aligned} & min \sum_{S} w_s x_s \\ \text{s.t.} \quad & \sum_{S: e \in S} x_s \geq c_e \quad \forall e \in E \\ & \sum_{e} c_e \geq p \cdot |E| \\ & 1 \geq c_e \geq 0 \qquad \forall e \in E \\ & x_s \geq 0 \qquad \forall S \in \{S_1, ..., S_m\} \end{aligned}$$

where variable c_e measures the coverage of element e.

Consider an algorithm for partial cover which randomly rounds the solution of linear program 1 (the one for real set cover). Prove that this algo1rithm constructs a solution of value no more than $\left(1 + \ln\left(\frac{1}{1-p}\right)\right) \cdot OPT_{LP}^{SC}$, where OPT_{LP}^{SC} is the value of the optimal solution to the set cover linear program.

Solution

We are interested to prove that the randomized rounding on LP for real set cover is an approximate solution for partial cover with optimal value $O(1 + \ln(\frac{1}{1-p})OPT_{LP}^{SC})$ where OPT_{LP}^{SC} is the solution of the LP for the set cover problem. Let be:

- $\beta = 1 + \ln\left(\frac{1}{1-p}\right)$;
- $\alpha > 1$ a costant;
- $\delta > 1$ a costant;
- SOL is the solution obtained by the union of some iterated randomized roundings;
- N number of elements.

We are interestend in prove that

$$P\big(cost(SOL) \leq \delta\beta \cdot OPT_{LP}^{SC} \wedge SOL \text{ is feasible}\big) \geq \frac{1}{2}$$

or

$$1 - P\big(cost(SOL) > \delta\beta \cdot OPT_{LP}^{SC}\big) - P\big(SOL \text{ is not feasible}\big) \geq \frac{1}{2}$$

If we randomize each variable x_i to 1 with probability x_i^* and we iterate the process $\alpha\beta$ times, it follows that:

$$\mathbb{E}\big(cost(SOL)\big) \leq \alpha\beta \cdot OPT_{LP}^{SC}$$

By Markov Inequality we have:

$$P\big(cost(SOL) > \delta\beta \cdot OPT_{LP}^{SC}\big) \leq \frac{\mathbb{E}\big(cost(SOL)\big)}{\delta\beta \cdot OPT_{LP}^{SC}}$$

$$P\big(cost(SOL) \geq \delta\beta \cdot OPT_{LP}^{SC}\big) \leq \frac{\mathbb{E}\big(cost(SOL)\big)}{\delta\beta \cdot OPT_{LP}^{SC}} \leq \frac{\alpha\beta \cdot OPT_{LP}^{SC}}{\delta\beta \cdot OPT_{LP}^{SC}} = \frac{\alpha}{\delta}$$

We can define:

- a random variable X_j such that $X_i = \begin{cases} 1 & \text{if element } i \text{ is not covered} \\ 0 & \text{else} \end{cases}$
- $X = \sum_{j \in U} X_i$ is the random variable equal to the number of covered elements;

$$\Rightarrow P\big(SOL \text{ is not feasible}\big) = P\big(X > N(1-p)\big)$$
$$P\big(X_i = 1\big) = P\big(\text{A given element is covered}\big)$$

Supposing that e_j is in k sets and is not covered only if in the $\alpha\beta$ iteration is never taken:

$$P(X_i = 1) = \left[\prod_{j: e_i \in S_j} (1 - x_j^*)\right]^{\alpha\beta} \leq \left[\left(1 - \frac{\sum_{j: e_i \in S_j}(x_j^*)}{k}\right)^k\right]^{\alpha\beta} \leq \left[\left(1 - \frac{1}{k}\right)^k\right]^{\alpha\beta} \leq \left[\frac{1}{e}\right]^{\alpha\beta}$$

Follows that (using again the Markov Inequality):

$$P\big(X > N(1-p)\big) \leq P\big(X \geq N(1-p)\big) \leq \frac{\mathbb{E}(X)}{N(1-p)} =$$
$$= \frac{\sum_{i=1}^{N} \mathbb{E}(X_i)}{N(1-p)} = \frac{\sum_{i=1}^{|U|} P(X_i = 1)}{N(1-p)} = N\left[\frac{1}{e}\right]^{\alpha\beta} \frac{1}{N(1-p)} =$$
$$N\left[\frac{1-p}{e}\right]^{\alpha} \frac{1}{N(1-p)} = \frac{(1-p)^{\alpha-1}}{e^{\alpha}} \leq \frac{1}{e^{\alpha}}$$

Finally:

$$P\big(cost(SOL) \leq \delta\beta \cdot OPT_{LP}^{SC} \wedge SOL \text{ is feasible}\big) \geq 1 - \frac{\alpha}{\delta} - \frac{1}{e^{\alpha}} \geq \frac{1}{2}$$

that for $\delta = 4\alpha$ and any $\alpha \geq 2$ we obtain $\frac{1}{e^{\alpha}} \leq \frac{1}{4}$. It follows:

$$P\big(cost(SOL) \leq \delta\beta \cdot OPT_{LP}^{SC} \wedge SOL \text{ is feasible}\big) \geq \frac{1}{2}$$