Correlated Equilibria: New Year's Eve in the lab

It is the last day of the year and, in the lab, there are some tasks $T_1, ..., T_m$ that should be performed before to run at home for the New Year's Eve party. Each task can be performed only by people with specific skills. Thus, for each worker i in the lab there is a subset $S_i \subset \{T_1, ..., T_m\}$ of tasks that she can perform. Let $A = (A_1, ..., A_n) \in S_1 \times ... \times S_n$ be an assignment of tasks to the n workers of the lab. Each worker i would like to leave the lab as soon as possible in order to join the New Year's Eve party. Thus each worker i has a cost $c_i(A)$ for the assignment A, representing the time that i takes to perform the assigned job, where the time depends on the number of people assigned to the same task and their skills. The aim of the worker is to minimize this cost. The workers discuss a lot about how to assign the tasks to themselves, but they are unable to find a solution that satisfies each of them. The head of the lab then propose the following solution: He will announce a probability distribution π on the possible assignments. Then he will draw a specific assignment A^* according to this distribution and suggests to each worker i the task A_i^* that he must perform (but worker i will not know which task has been suggested to other people). The worker i will then perform the task that minimizes the expected cost, where the expectation is taken with respect to the announced distribution π conditioned on the task assigned to i being A_i^*. Prove that the head of the lab can always compute in polynomial time a distribution π such that each worker i will perform the suggested task.

Solution

Let be:

- N = number of workers;
- S_i = tasks that each worker i can perform;
- $m_i = |S_i|$;
- $A = S_1 \times S_2 \times ... \times S_N$;
- $A_{-i} = S_1 \times S_2 \times ... \times S_{i-1} \times S_{i+1} \times ... \times S_N$;
- A_i^* = the suggested task to worker i;

The worker i will do the suggested work if and only if:

$$\sum_{a \in A_{-i}} c_i(a, A_i^*) P\big(a | A_i^*\big) \leq \sum_{a \in A_{-i}} c_i(a, a_i') P\big(a | A_i^*\big)$$

$$\sum_{a \in A_{-i}} c_i(a, A_i^*) \frac{P\big(a \cap A_i^*\big)}{P(A_i^*)} \leq \sum_{a \in A_{-i}} c_i(a, a_i') \frac{P\big(a \cap A_i^*\big)}{P(A_i^*)}$$

$$\sum_{a \in A_{-i}} c_i(a, A_i^*) P\big(a \cap A_i^*\big) \leq \sum_{a \in A_{-i}} c_i(a, a_i') P\big(a \cap A_i^*\big)$$

$\forall A_i^*, a_i' \in A_i$

It follows that for each couple $\langle a_i, a_i' \rangle$ we have some constraints.

$$\begin{cases} \sum_{a\in A_{-i}} c_i(a,a_i)P\big(a\cap A_i^*\big) \le \sum_{a\in A_{-i}} c_i(a,a_i')P\big(a\cap A_i^*\big) & \forall a_i, a_i' \in S_i, \forall i = 1, ..., N \\ \sum_{a\in A} P(a) = 1 & \forall a \in A \\ P(a) \ge 0 \end{cases}$$

This linear programming:

- has a finite number of constraints: $\sum_{i=1}^{N} m_i^2$
- has a finite number of variables: $|A|$
- is always feasible because:
  - the game is finite (finite number of players, finite number of states and finite number of constraints)
  - every finite game has a *Nash Equilibrium*
  - every *Correlated Equilibrium* is a *Nash Equilibrium*
  - the proposed problem consist in finding *Correlated Equilibrium*

It follows that in polynomial time the laboratory's head can find the required distribution of probability.